SOME HEAT CONDUCTION PROBLEMS IN AN INHOMOGENEOUS BODY

R. B. Nudel'man

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Solutions are obtained for the heat conduction equation in the case when the thermal conductivity is a homogeneous function of the coordinates.

Many problems of mathematical physics reduce to integration of the following differential equation:

$$A\frac{\partial\varphi}{\partial t} = \operatorname{div}(K \operatorname{grad}\varphi) + B\varphi + C, \qquad (1)$$

where the coefficients A, B, C, and K and the desired function φ depend on all the coordinates and on time.

The majority of solutions of Eq. (1) published in the literature has been obtained under conditions where A, B, C, and K are constants.

The objective of the present paper is to determine exact particular solutions of (1) under the condition where less rigorous restrictions are imposed on the coefficients of the equation.

It is known [1] that integration of (1) does not present special difficulty if the desired function and the coefficients depend on one space variable, i.e., when (1) has the form

$$A(r,t)\frac{\partial\varphi}{\partial t} = \frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n K(r,t)\frac{\partial\varphi}{\partial r}\right) + B(r,t)\varphi + C(r,t), \qquad (2)$$

where n = 0 for r = x; n = 1 for $r = (x^2 + y^2)^{1/2}$ and n = = 2 for $r = (x^2 + y^2 + z^2)^{1/2}$.

We shall show that Eq. (1) reduces to an equation analogous to (2), not only when the coefficients A, B, C, and K depend only on r and t, but also in the more general case.

Let coefficients A, B, and C be represented by the following relations:

$$A = a(r,t)K; \quad B = b(r,t)K; \quad C = c(r,t)K, \qquad (3)$$

where K = K(x, y, z) is a homogeneous function of degree m,

$$xK_x + yK_y + zK_z = mK(x,y,z). \qquad (4)$$

If $\varphi = \varphi(r,t)$, the operator

$$L(\varphi) = \operatorname{div}(K \operatorname{grad}\varphi) \qquad (5)$$

reduces under condition (4) to the form

$$L(\varphi) = \left(\frac{\partial^2\varphi}{\partial r^2} + \frac{m+2}{r}\frac{\partial\varphi}{\partial r}\right)K(x,y,z). \qquad (6)$$

Taking account of (3)–(6), we obtain the equation

$$a\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial r^2} + \frac{m+2}{r}\frac{\partial\varphi}{\partial r} + b\varphi + c. \qquad (7)$$

Equation (7) does not depend on the specific form of the function K(x, y, z), but only on the exponent of the homogeneity. For example, we may take the function K in the form

$$K(x,y,z) = \sum_{i,j,k} a_{ijk}x^i y^j z^k, \quad i+j+k = m, \quad i,j,k = 0,1,2\ldots, \qquad (8)$$

where a_{ijk} are arbitrary constants.

The final number of constants a_{ijk} does not permit a full description of any inhomogeneous medium, but since the exponent of homogeneity m may be any real number, it may be taken to be large enough to allow the inhomogeneity of the medium to be taken into account with sufficient accuracy in a number of cases.

Let $(\partial\varphi/\partial t) = 0$ (a stationary problem) and B = C = = 0. Then Eq. (7) takes the form

$$\frac{d^2\varphi}{dr^2} + \frac{m+2}{r}\frac{d\varphi}{dr} = 0. \qquad (9)$$

The solution of this equation will be a function depending on two arbitrary constants

$$\varphi(r) = C_1 r^{-(m+1)} + C_2. \qquad (10)$$

For m = 0 we obtain the well-known special solution of the Laplace equation, which is simultaneously also a solution of the equation

$$\frac{\partial}{\partial x}\left(K\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial\varphi}{\partial z}\right) = 0; \qquad (11)$$

K(x, y, z) is any homogeneous function of degree.

Example. We shall determine the radial heat flux in a hollow sphere, if the surface $r = r_1$ has temperature φ_1, and the surface $r = r_2$ has temperature φ_2. The thermal conductivity is a uniform function of degree m. These requirement are satisfied by the function

$$\varphi(r) = \frac{r_1^{m+1}r_2^{m+1}(\varphi_1 - \varphi_2)}{r_2^{m+1} - r_1^{m+1}}\left(\frac{1}{r^{m+1}} - \frac{1}{r_1^{m+1}}\right) + \varphi_1. \qquad (12)$$

It is interesting that the temperature distribution does not depend on the specific form of the thermal conductivity, but depends only on the exponent of uniformity.

The total heat flux is determined from the formula

$$Q = -\int_0^{2\pi} d\theta \int_0^{\pi} K(r,\Theta,\psi)\frac{\partial\varphi}{\partial r}r^2 \sin\psi\, d\psi. \qquad (13)$$

Hence, it may be seen that Q depends on the specific form of the thermal conductivity. If, for example,

$$K(x,y,z) = \alpha x^{2n} + \beta y^{2n} + \gamma z^{2n}, \quad n = 0,1,2,\ldots, \qquad (14)$$

where α, β, γ are arbitrary constants, then

$$Q = \frac{4\pi(\varphi_1 - \varphi_2) r_1^{2n+1} r_2^{2n+1}}{r_1^{2n+1} - r_2^{2n+1}} (\alpha + \beta + \gamma). \quad (15)$$

Various known methods of mathematical physics (the Fourier method, the method of eigenfunctions, etc.) may be used in integration of (7) in the general case.

If the coefficients a, b, and c of (7) are functions of r only, we may seek a solution in the form

$$\varphi(r, t) = u(r, t) + v(r), \quad (16)$$

where the functions u and v satisfy the equation

$$a(r)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{m+2}{r}\frac{\partial u}{\partial r} + b(r)u, \quad (17)$$

and

$$\frac{d^2 v}{dr^2} + \frac{m+2}{r}\frac{dv}{dr} + bv + c(r) = 0. \quad (18)$$

Separating the variables in (17), we obtain

$$u(r, t) = w(r)\exp(-\lambda t); \quad (19)$$

where λ is a constant, and w(r) is the solution of the equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{m+2}{r}\frac{\partial w}{\partial r} + (b + \lambda a) w = 0. \quad (20)$$

The substitution

$$w(r) = r^{-\frac{m+2}{2}}\psi(r) \quad (21)$$

reduces to the integration of the following equation

$$\frac{d^2\psi}{dr^2} + \left[b(r) - \frac{m(m+2)}{4r^2} + \lambda a\right]\psi = 0. \quad (22)$$

Wishing to obtain simpler eigenfunctions, we shall confine attention to the particular case of

$$a = a_0, \quad b(r) = \frac{1}{4}m(m+2)r^{-2}, \quad \lambda a_0 = \alpha_k^2, \quad (23)$$

where a_0 and α_k are arbitrary constants. Then the solution of (22) will be

$$\psi(r) = A_k \cos\alpha_k r + B_k \sin\alpha_k r. \quad (24)$$

The general solution of (18), under the condition that b(r) is determined by (23) and c(r) is any function of r, has the form

$$v(r) = \left(D_1 + \int r^{\frac{m+4}{2}} c(r)\, dr\right) r^{-\frac{m+2}{2}} + \left(D_2 - \int r^{\frac{m+2}{2}} c(r)\, dr\right) r^{-\frac{m}{2}}, \quad (25)$$

where D_1 and D_2 are arbitrary constants. In the special case when $c = c_0 r^n$, we obtain

$$v(r) = D_1 r^{-\frac{m+2}{2}} + D_2 r^{-\frac{m}{2}} - \frac{c_0}{\delta(\delta-1)} r^{n+2};$$
$$2\delta = m + 2n + 6. \quad (26)$$

Because of the linearity of Eq. (7), the solutions obtained are additive. Finally we obtain

$$\varphi(r, t) = \left(D_1 + \int r^{\frac{m+4}{2}} c(r)\, dr\right) r^{-\frac{m+2}{2}} + \left(D_2 - \int r^{\frac{m+2}{2}} c(r)\, dr\right) r^{-\frac{m}{2}} + r^{-\frac{m+2}{2}} \sum_k (A_k \cos\alpha_k r + B_k \sin\alpha_k r)\exp\left(-\frac{\alpha_k^2}{a_0} t\right). \quad (27)$$

The arbitrary constants D_1, D_2, A_k and B_k must be determined from the boundary and initial conditions.

As an example we shall examine a spherical shell, on the outer surface of which a constant temperature of zero is maintained, and a constant temperature T on the internal surface. This leads to the following boundary conditions:

$$\varphi = 0, \quad r = r_1;$$
$$\varphi = T, \quad r = r_2. \quad (28)$$

With $c = c_0 r^n$, the solution

$$\varphi(r, t) = D_1 r^{-\frac{m+2}{2}} + D_2 r^{-\frac{m}{2}} - \frac{c_0}{\delta(\delta-1)} r^{n+2} + r^{-\frac{m+2}{2}} \sum_k M_k \sin\alpha_k = (r - r_2)\exp\left(-\frac{\alpha_k^2}{a_0} t\right) \quad (29)$$

satisfies these boundary conditions, where

$$\alpha_k = \frac{k\pi}{r_1 - r_2}, \quad \delta \neq 0, 1,$$
$$D_1 = \frac{1}{\delta(\delta-1)} r_1^{\delta} - D_2 r_1,$$
$$D_2 = \frac{T r_2^{\frac{m+2}{2}} + \frac{1}{\delta(\delta-1)}(r_2^{\delta} - r_1^{\delta})}{r_2 - r_1}. \quad (30)$$

The constant coefficient M_k must be determined from the initial condition

$$\varphi|_{t=0} = f(r). \quad (31)$$

The problem of determining M_k from condition (31) does not present any particular difficulty.

If we put m and c_0 equal to zero in (29) (this implies that b and c are equal to zero in (7)) and take the initial condition in the form

$$\varphi|_{t=0} = T, \quad (32)$$

we obtain the well-known solution of the equation

$$a_0 \frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r}, \quad (33)$$

which has the form

$$\varphi(r, t) = \frac{T r_2}{r_1 - r_2}\left(\frac{r_1}{r} - 1\right) - \frac{2 r_1 T}{\pi} \sum_{k=1,2,3}^{\infty} \frac{\cos k\pi}{kr} \times \sin\alpha_k (r - r_2)\exp\left(-\frac{\alpha_k^2}{a_0} t\right). \quad (34)$$

We note that in the case of two dimensions ($r = (x^2 + y^2)^{1/2}$), m + 2 should be replaced by m + 1 in Eq. (7).

For a region with cylindrical symmetry, we are interested in a solution of the form

$$\varphi = \varphi(\rho, z, t), \quad \rho = \sqrt{x^2 + y^2}. \qquad (35)$$

We shall impose the following restrictions on the coefficients A, B, C and K:

$$A = a(\rho, z, t)K; \; B = b(\rho, z, t)K; \quad C = c(\rho, z, t)K;$$
$$xK_x + yK_y = mK(x, y). \qquad (36)$$

The operator $L(\varphi)$ in this case takes the form

$$L(\varphi) = \frac{\partial^2 \varphi}{\partial \rho^2} + \frac{m+1}{\rho}\frac{\partial \varphi}{\partial \rho} + \frac{\partial^2 \varphi}{\partial z^2}, \qquad (37)$$

while Eq. (1) reduces to the following:

$$a\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial \rho^2} + \frac{m+1}{\rho}\frac{\partial \varphi}{\partial \rho} + \frac{\partial^2 \varphi}{\partial z^2} + b\varphi + c. \qquad (38)$$

Although this equation is simpler than the original, it is, however, sufficiently complicated that it cannot be integrated for any a, b, and c. Bearing in mind that we wish to apply the method of separation of variables later on, we shall consider that a and b are functions of only the variable z.

In order to eliminate the variable t, we may apply a Laplace transformation

$$\int_0^\infty \exp(-pt)\,\varphi(\rho, z, t)\,dt = p\bar{\varphi}(\rho, z) - \bar{\varphi}_0. \qquad (39)$$

Then Eq. (38), for $\bar{\varphi}_0 = 0$ and c = 0, takes the form

$$a(z)\,p\bar{\varphi} = \frac{\partial^2 \bar{\varphi}}{\partial \rho^2} + \frac{m+1}{\rho}\frac{\partial \bar{\varphi}}{\partial \rho} + \frac{\partial^2 \bar{\varphi}}{\partial z^2} + b(z)\bar{\varphi}. \qquad (40)$$

Let $\bar{\varphi}(\rho, z) = f(\rho)\psi(z)$; then it follows from (40) that

$$\left(\frac{d^2 f}{d\rho^2} + \frac{m+1}{\rho}\frac{df}{\rho}\right)\Big/ f + \left(\frac{d^2\psi}{dz^2} + [b - pa]\psi\right)\Big/ \psi = 0. \qquad (41)$$

Hence to determine $f(\rho)$ and $\psi(z)$ we obtain two ordinary differential equations:

$$\frac{d^2\psi}{dz^2} + \mu(z)\psi = 0, \quad \mu(z) = b(z) - pa(z) = \lambda^2 \qquad (42)$$

and

$$\frac{\partial^2 f}{d\rho^2} + \frac{m+1}{\rho}\frac{df}{d\rho} \pm \lambda^2 f = 0. \qquad (43)$$

The solution of (43), depending on the sign of λ^2, has the form

$$f(\rho) = \rho^{-m/2}\begin{cases} Z_{m/2}(\lambda\rho) \\ Z_{m/2}(i\lambda\rho) \end{cases}, \qquad (44)$$

where $Z_\nu(x)$ is a linear combination of Bessel functions of the first and second kind, of imaginary or real argument.

A large number of exact solutions is known for Eq. (42) for specific form of the coefficient $\mu(z)$.

Because of the linearity of (40), the solution obtained may be added, which allows solution of quite a large class of boundary problems.

REFERENCE

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Advanced Control Engineering College, Khar'kov